CS 4100 // artificial intelligence

instructor: byron wallace

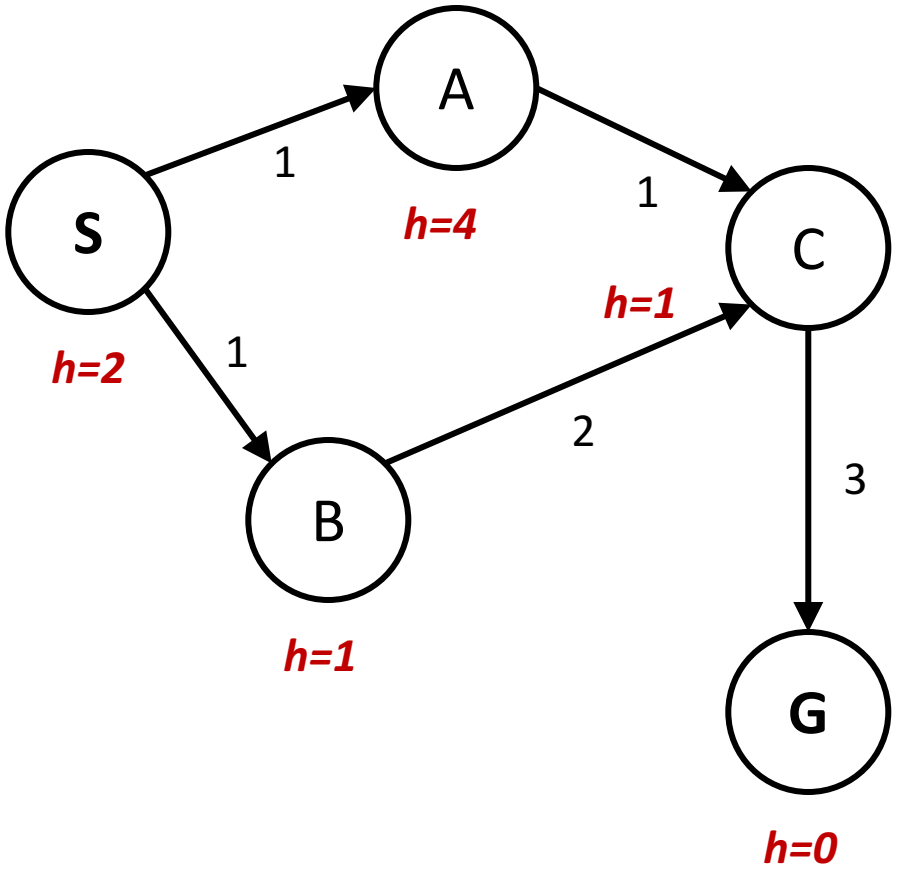
Constraint Satisfaction Problems

Attribution: many of these slides are modified versions of those distributed with the *UC Berkeley CS188* materials
Thanks to John DeNero and Dan Klein

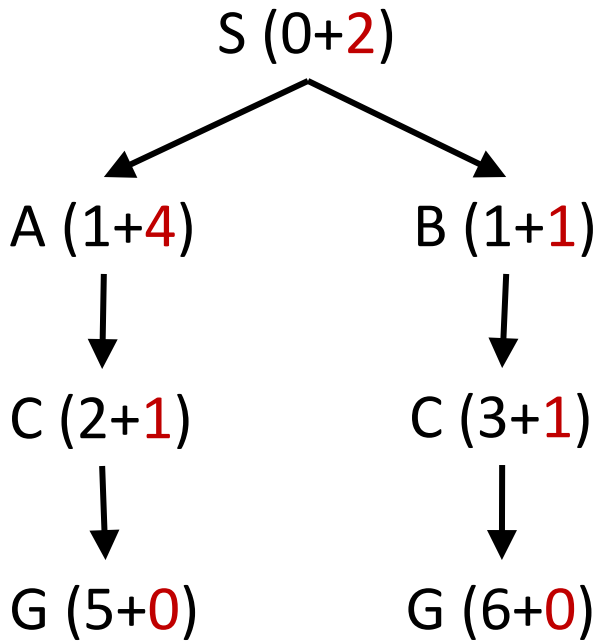
But first: wrapping up A* from last time

A* Graph search gone wrong?

State space graph

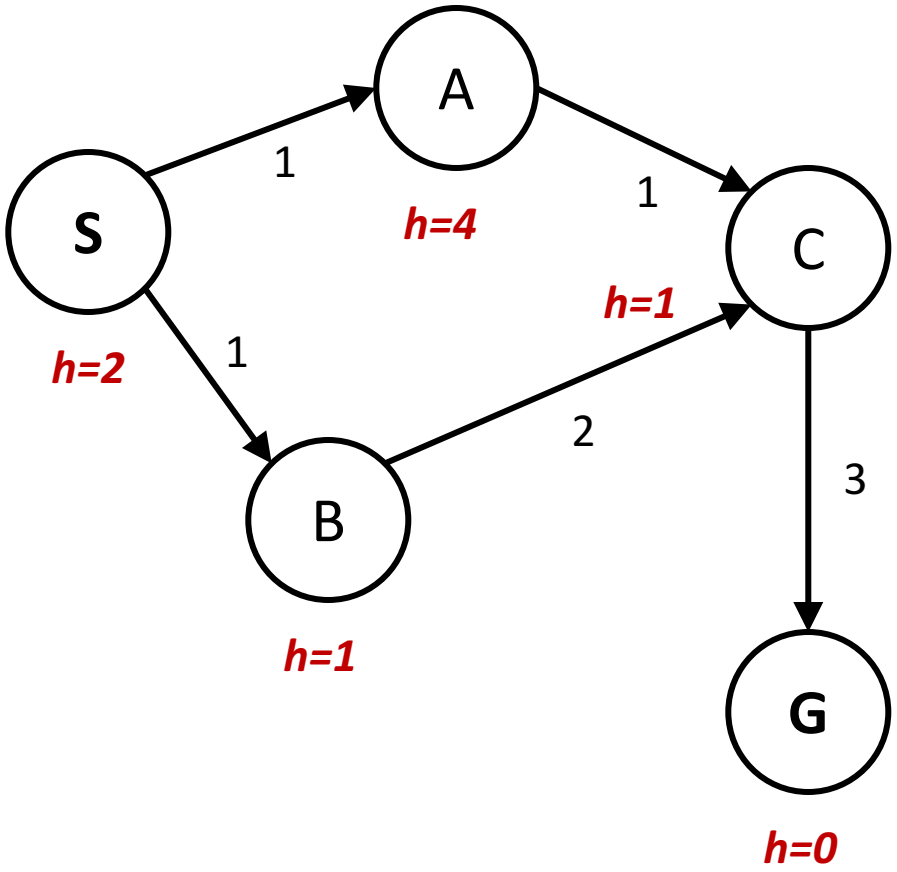


Search tree

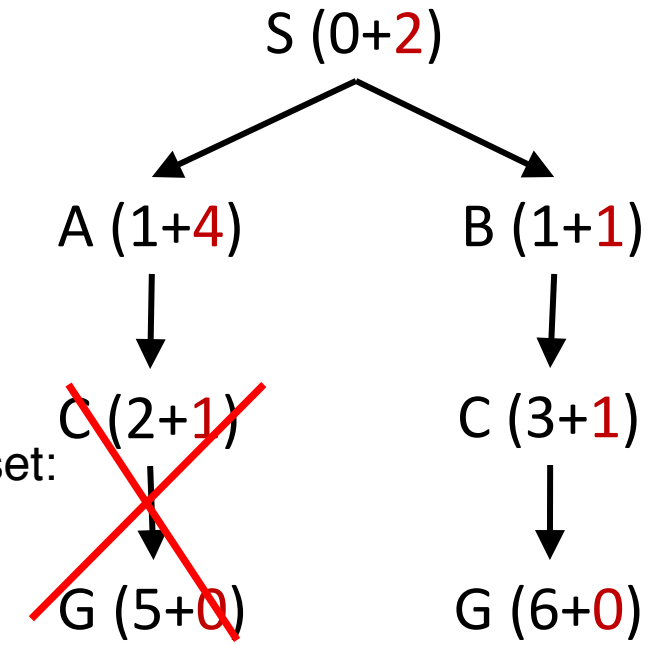


A* Graph search gone wrong?

State space graph



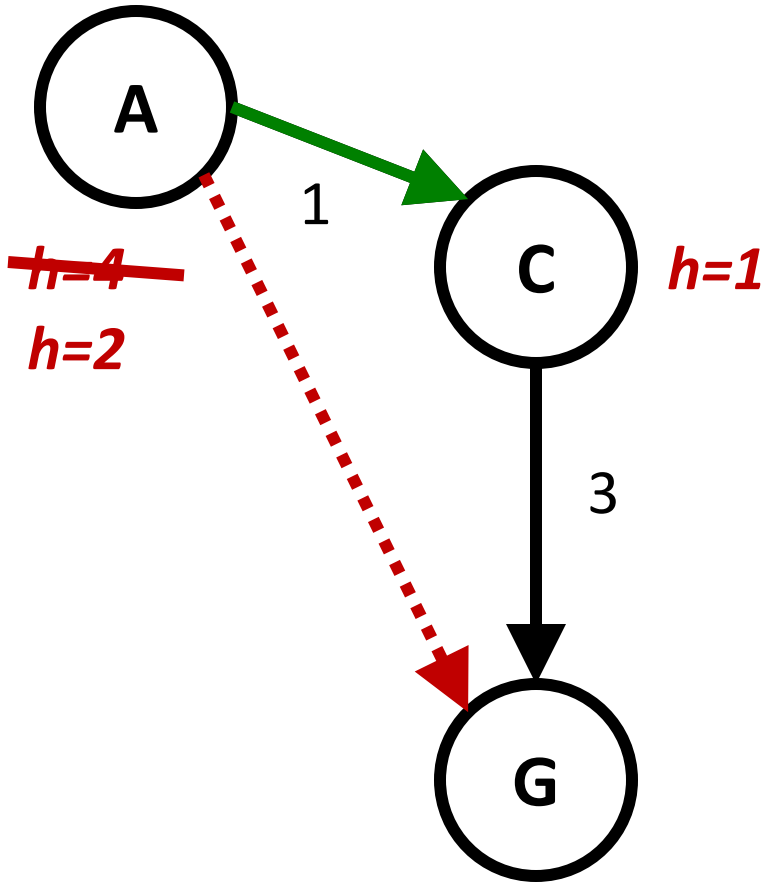
Search tree



Closed set:
{*SBC*}

Whoops! What went wrong??

Consistency of heuristics



Main idea: estimated heuristic costs ≤ actual costs

- **Admissibility**: heuristic cost ≤ actual cost to goal

  h(A) ≤ actual cost from A to G

- **Consistency**: heuristic "arc" cost ≤ actual cost for each arc

  h(A) – h(C) ≤ cost(A to C)

Consequences of consistency:

The f value along a path never decreases

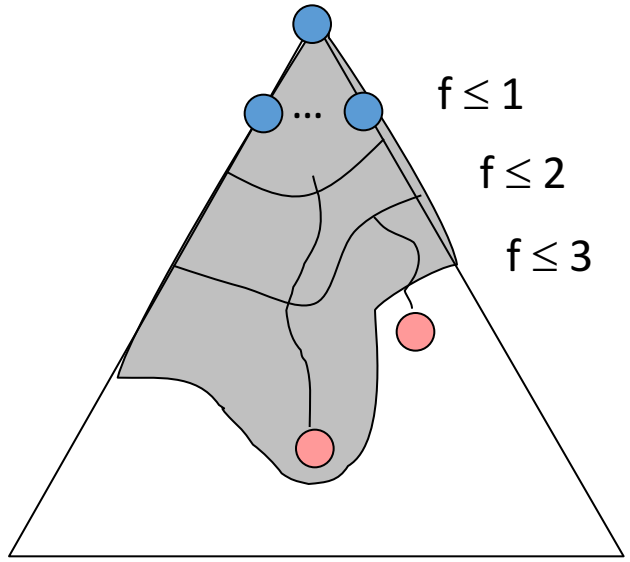
h(A) ≤ cost(A to C) + h(C)

A* graph search is optimal

Optimality of A* graph search

Sketch: consider what A* does with a consistent heuristic:

- Fact 1: In tree search, A* expands nodes in increasing total f value (f-contours)
- Fact 2: For every state s, nodes that reach s optimally are expanded before nodes that reach s suboptimally
- Result: A* graph search is optimal



Optimality

Tree search:

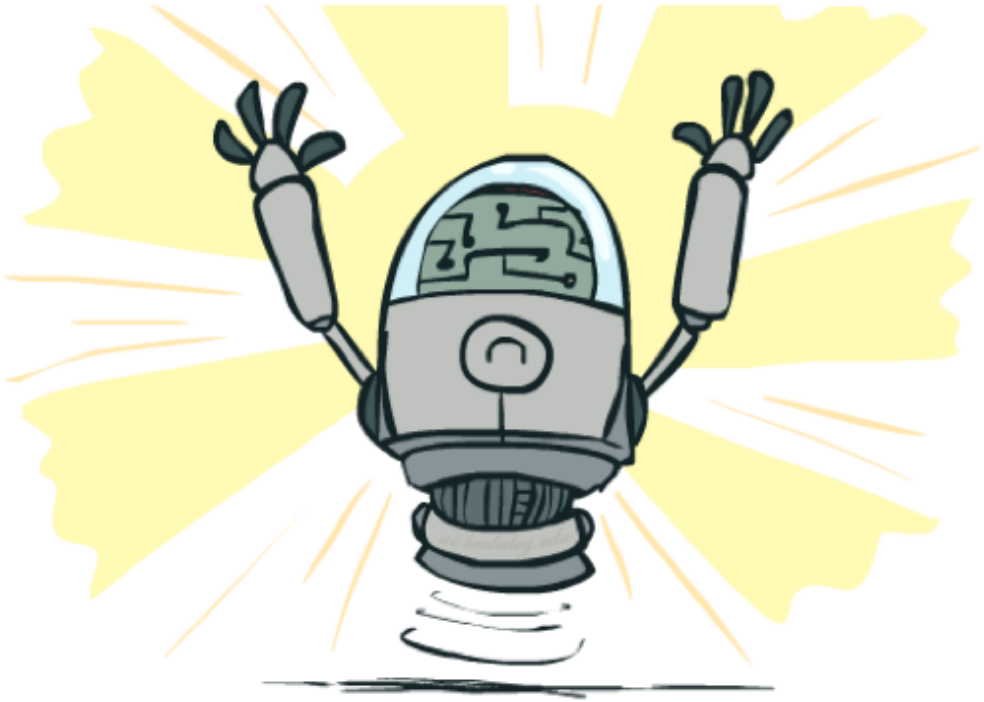
- A* is optimal if heuristic is admissible
- UCS is a special case (h = 0)

Graph search:

- A* optimal if heuristic is consistent
- UCS optimal (h=0 is consistent)

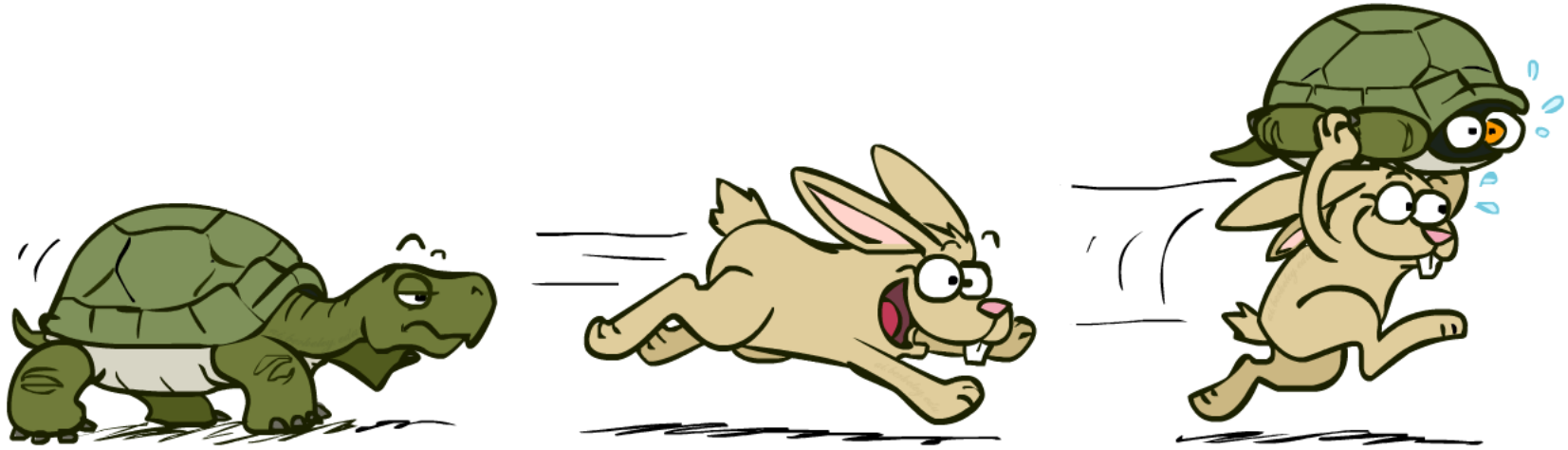
Consistency implies admissibility

In general, most natural admissible heuristics tend to be consistent, especially if from relaxed problems



A* Summary

- A* uses both backward costs and (estimates of) forward costs
- A* is optimal with admissible / consistent heuristics
- Heuristic design is key: often use relaxed problems



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Constraint Satisfaction Problems

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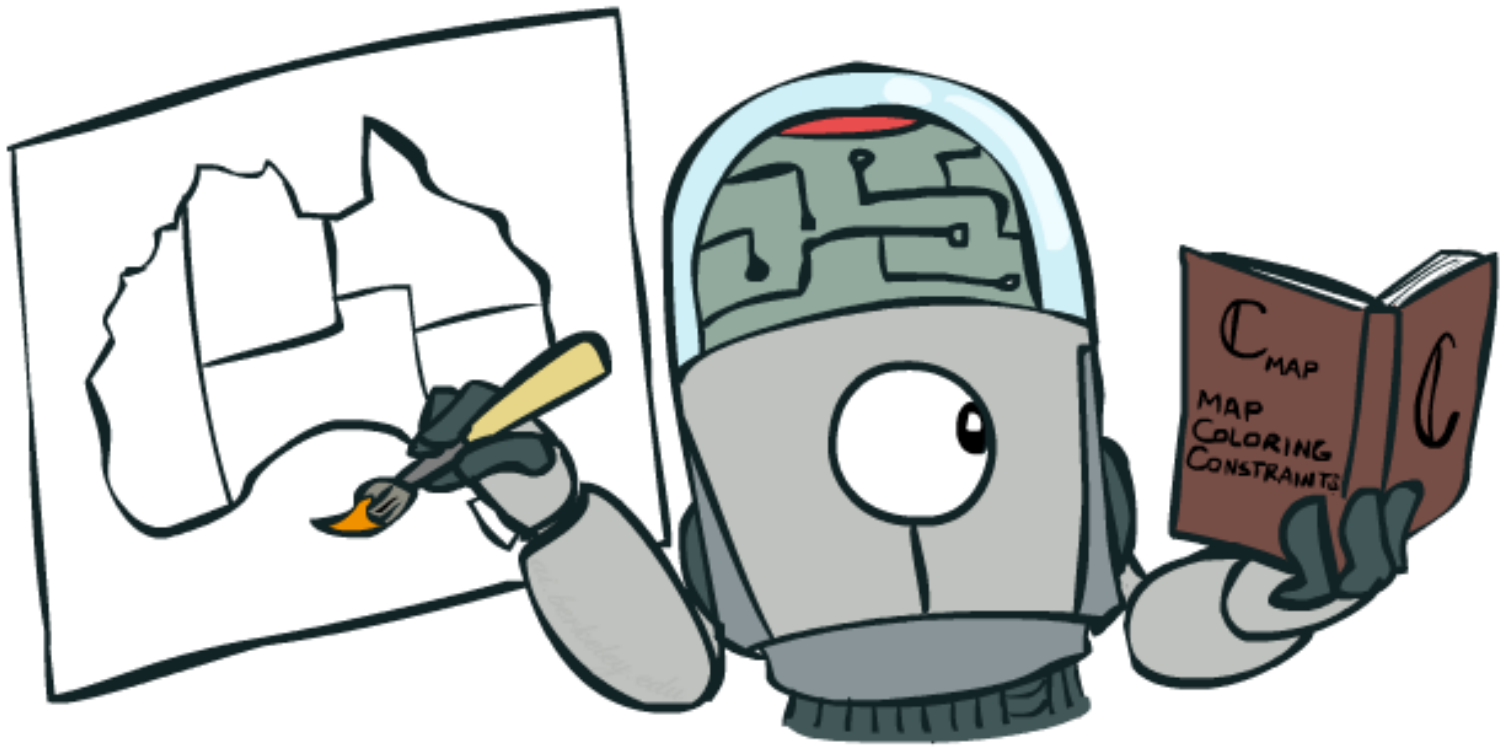
Search, so far

- Assumptions we have made about the world: a single agent, deterministic actions, fully observed state, discrete state space.

- So far we have mostly cared about *planning* or finding sequences of actions
  - The path to the goal is the important thing
  - Paths have various costs, depths
  - Heuristics give problem-specific guidance

- Other problems concern *identification*: finding assignments for variables
  - The goal itself is important, ***not the path***
  - All paths at the same depth (for some formulations)
  - *Constraint Satisfaction Problems (CSPs)* are specialized for identification problems

Constraint satisfaction problems



Constraint satisfaction problems

Standard search problems:

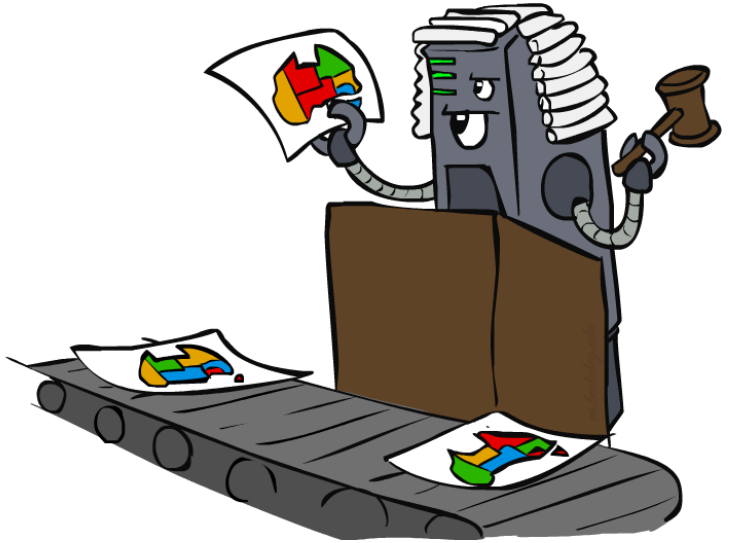
- State is a “black box”: arbitrary data structure
- Goal test can be any function over states
- Successor function can also be anything

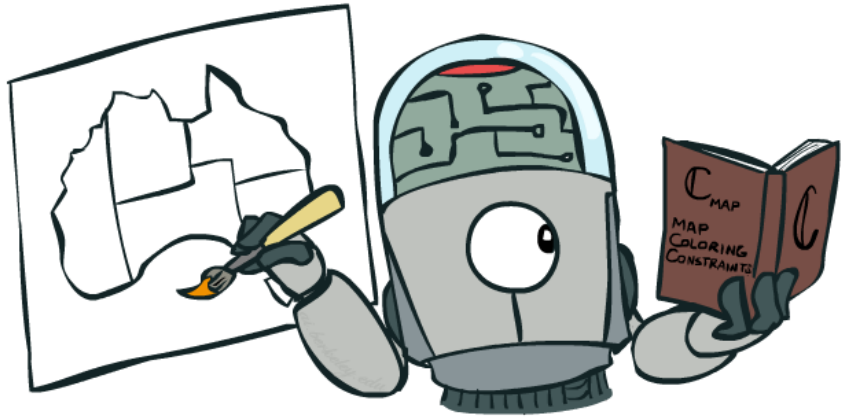
Constraint satisfaction problems (CSPs):

- A special subset of search problems
- State is defined by variables X_i with values from a domain D (sometimes D depends on i)
- *Goal test* is a set of constraints specifying allowable combinations of values for subsets of variables

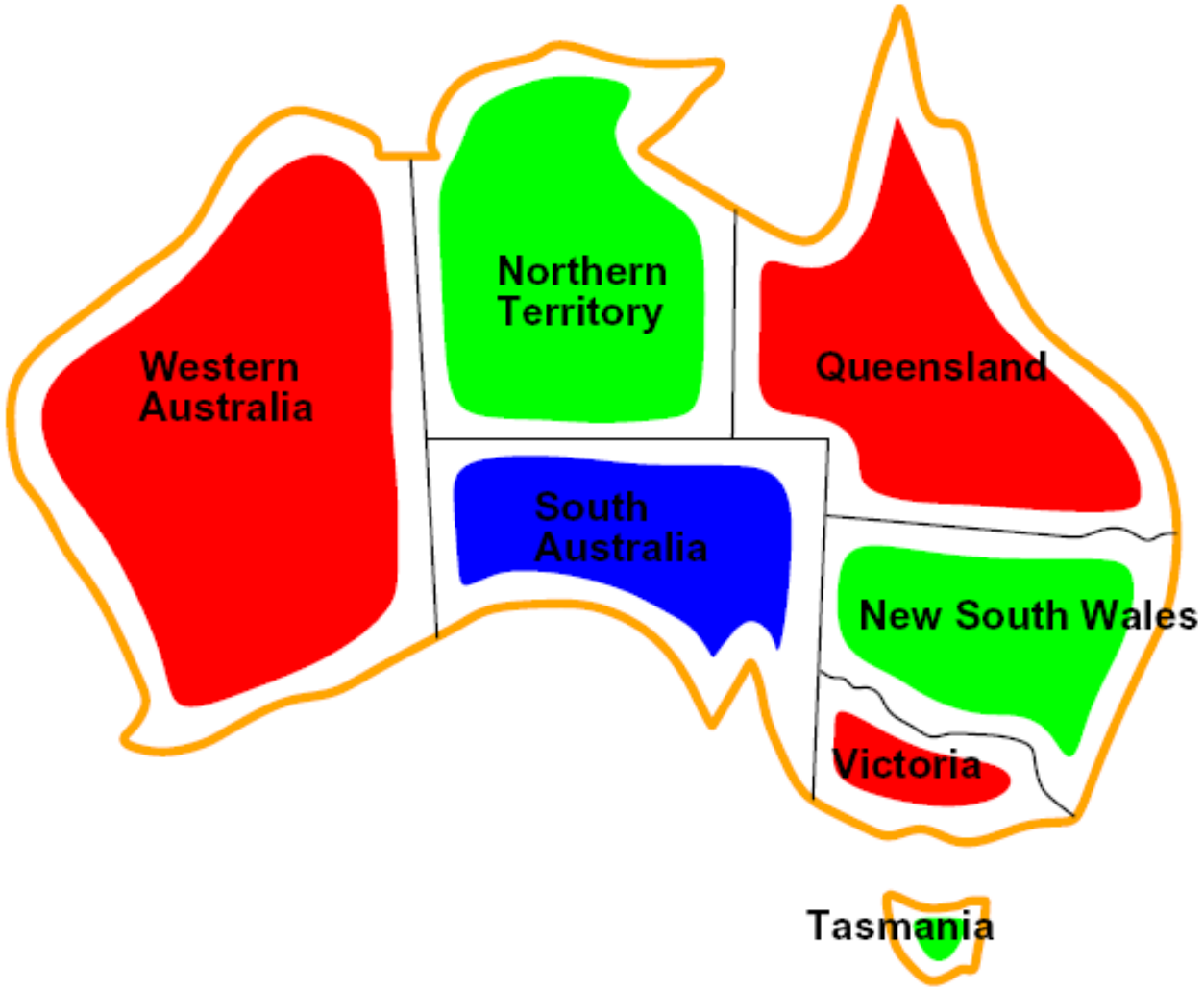
Simple example of a *formal representation language*

Allows useful general-purpose algorithms with more power than standard search algorithms





CSP Examples



Map coloring

Variables: $\mathrm{WA}, \mathrm{NT}, \mathrm{Q}, \mathrm{NSW}, \mathrm{V}, \mathrm{SA}, \mathrm{T}$

Domain: $D = \{\mathrm{red}, \mathrm{green}, \mathrm{blue}\}$

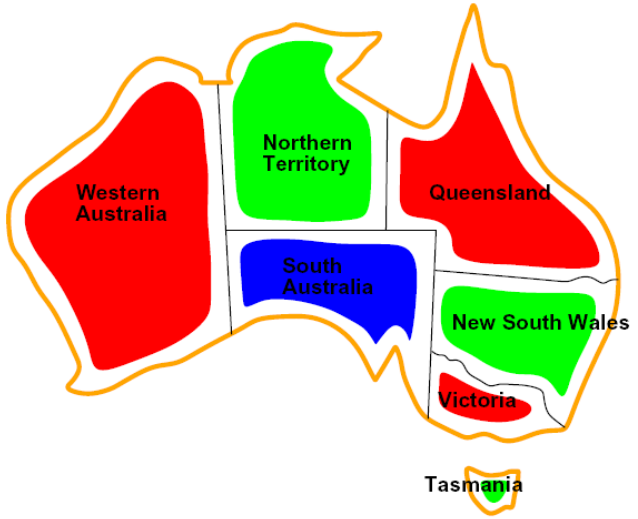
Constraints: adjacent regions must have different colors

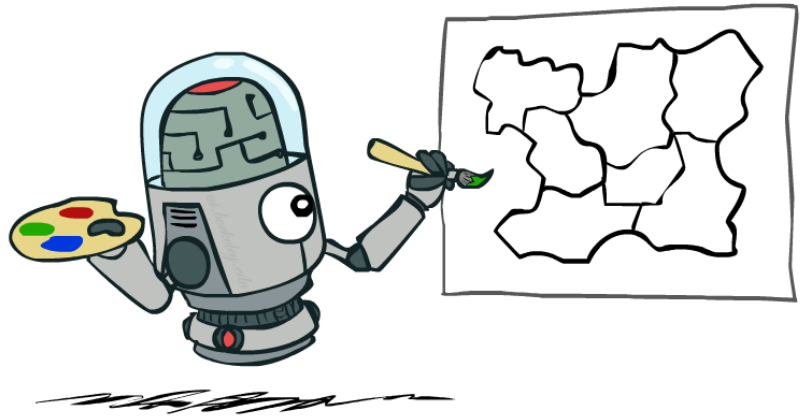
Implicit: $\mathrm{WA} \neq \mathrm{NT}$

Explicit: $(\mathrm{WA}, \mathrm{NT}) \in \{(\mathrm{red}, \mathrm{green}), (\mathrm{red}, \mathrm{blue}), \ldots\}$

Solutions are *assignments satisfying all constraints*, e.g.:

$\{\mathrm{WA}{=}\mathrm{red}, \mathrm{NT}{=}\mathrm{green}, \mathrm{Q}{=}\mathrm{red}, \mathrm{NSW}{=}\mathrm{green}, \mathrm{V}{=}\mathrm{red}, \mathrm{SA}{=}\mathrm{blue}, \mathrm{T}{=}\mathrm{green}\}$

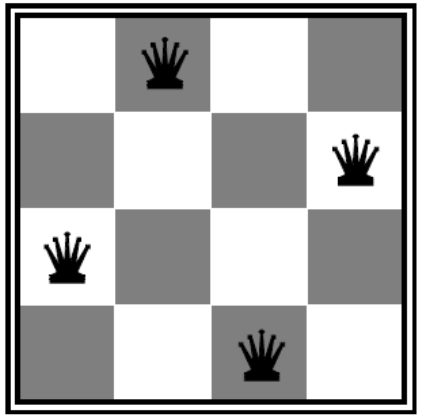


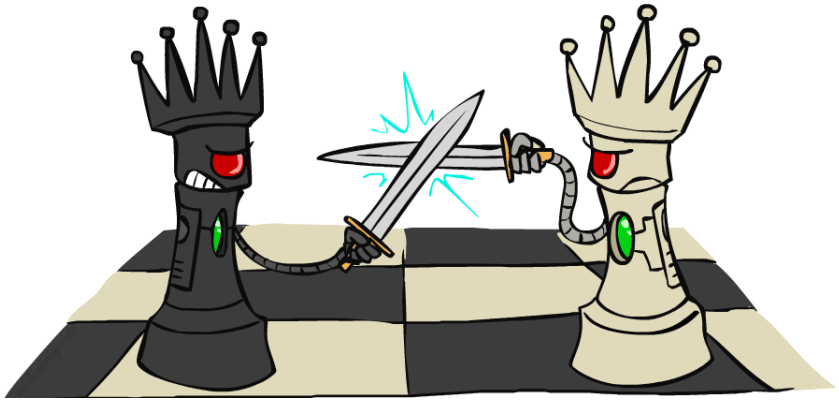


N-Queens

Formulation 1

- Variables: X_{ij}
- Domains: $\{0, 1\}$
- Constraints





$$\forall i, j, k \quad (X_{ij}, X_{ik}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \quad (X_{ij}, X_{kj}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \quad (X_{ij}, X_{i+k,j+k}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \quad (X_{ij}, X_{i+k,j-k}) \in \{(0,0), (0,1), (1,0)\}$$

$$\sum_{i,j} X_{ij} = N$$

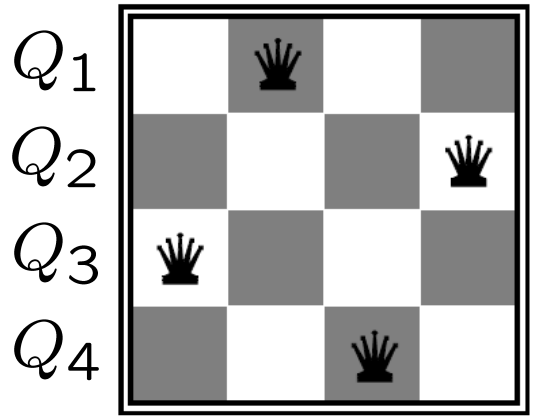
N-Queens

Formulation 2

- Variables: Q_k
- Domains: $\{1, 2, 3, \ldots N\}$
- Constraints:

  Implicit: $\forall i, j \;\; \text{non-threatening}(Q_i, Q_j)$

  Explicit: $(Q_1, Q_2) \in \{(1, 3), (1, 4), \ldots\}$



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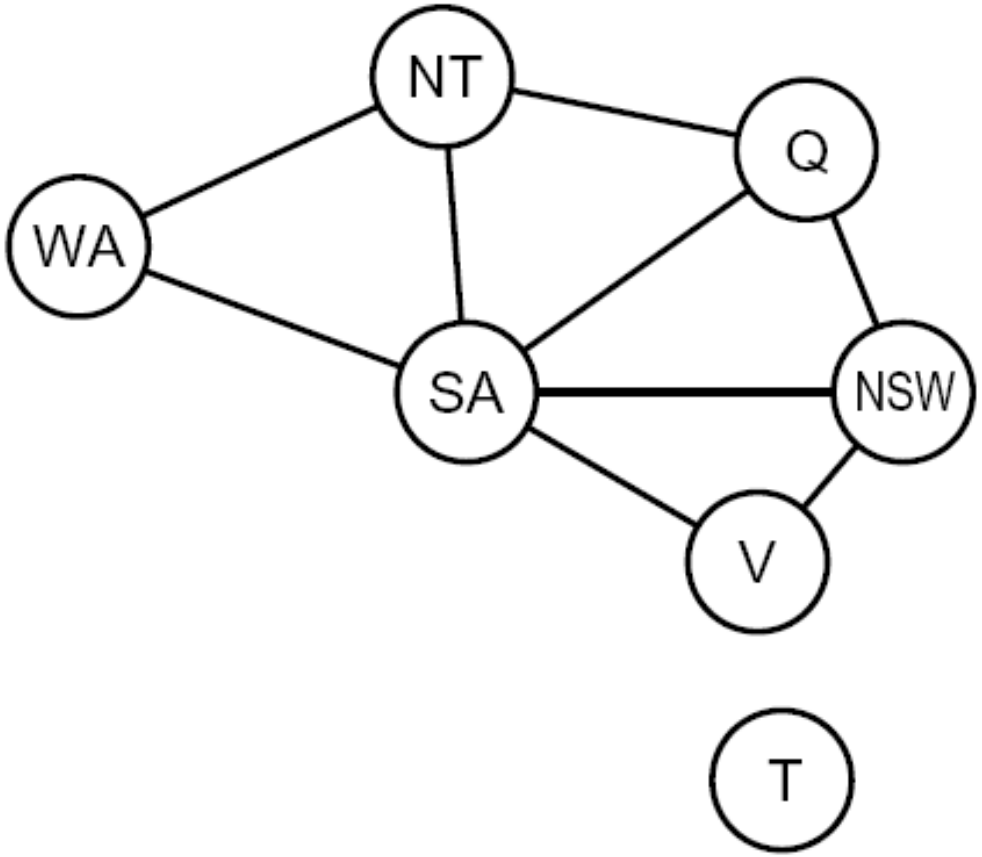
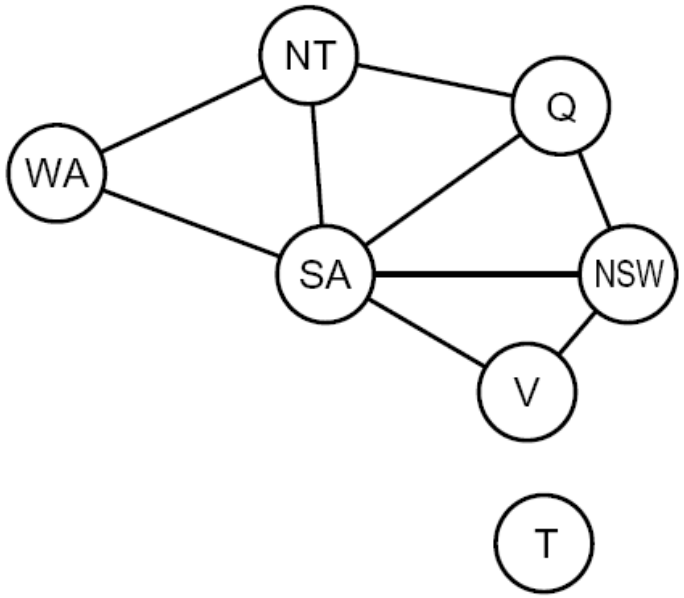
Constraint graphs



Constraint graphs

- Binary CSP: each constraint relates (at most) two variables
- Binary constraint graph: nodes are variables, arcs show constraints
- General-purpose CSP algorithms use the graph structure to speed up search. E.g., Tasmania is an *independent* subproblem!



Cryptarithmetic

Variables:

$$F \; T \; U \; W \; R \; O \; X_1 \; X_2 \; X_3$$

Domains:

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

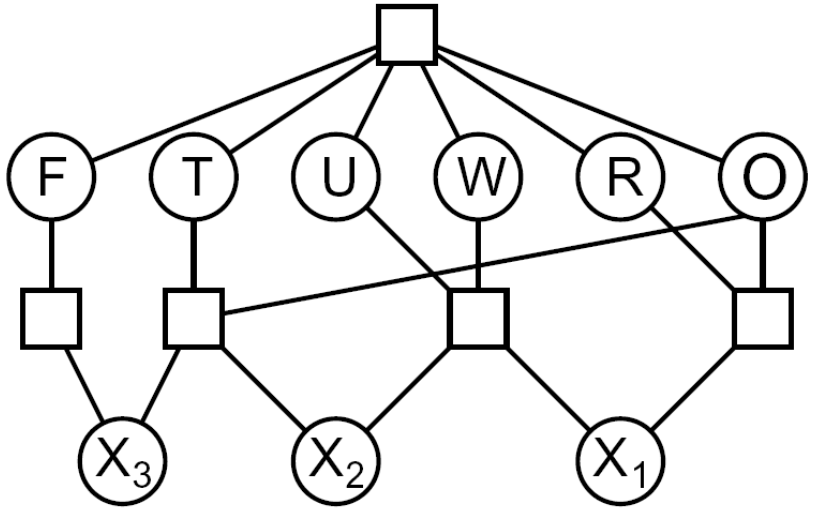
Constraints:

$$\text{alldiff}(F, T, U, W, R, O)$$

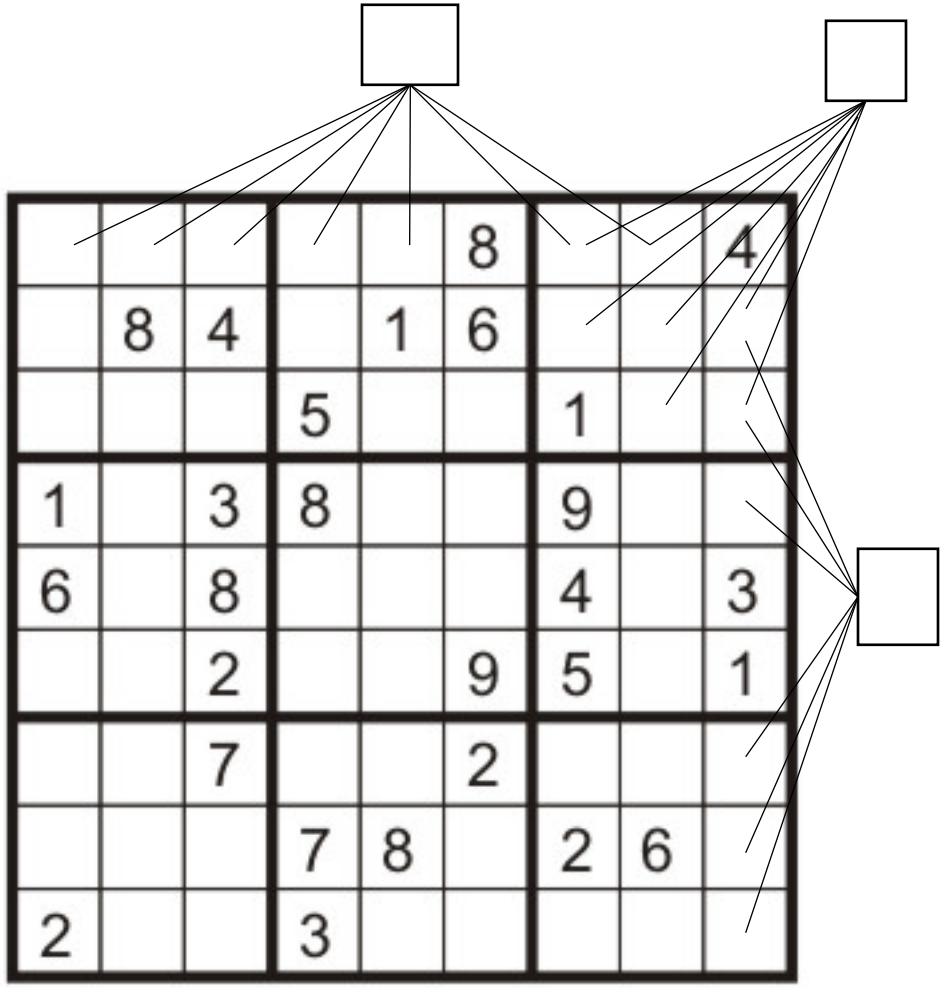
$$O + O = R + 10 \cdot X_1$$

$$\cdots$$

```
  T W O
+ T W O
-------
F O U R
```



Sudoku



Variables:

Each (open) square

Domains:

{1,2,…,9}

Constraints:

9-way all-diff for each column

9-way all-diff for each row

9-way all-diff for each *region*

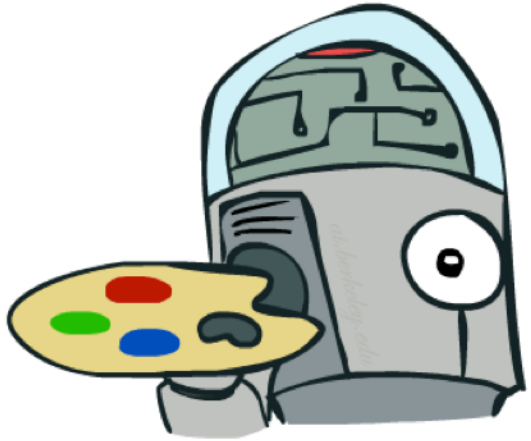
(or can have a bunch of pairwise inequality constraints)

Varieties of CSPs and constraints

Varieties of CSPs

Discrete Variables

- Finite domains
  - n variables with domain size d means $O(d^n)$ complete assignments
  - E.g., Boolean CSPs, including Boolean satisfiability (NP-complete)



- Infinite domains (integers, strings, etc.)
  - e.g., job scheduling, variables are start/end times for each job
  - Linear constraints solvable, nonlinear undecidable

Continuous variables

- E.g., start/end times for Hubble Telescope observations
- Linear constraints solvable in polynomial time by Linear Programming methods (we won't cover this here, but a huge and important topic)

Varieties of constraints

Varieties of Constraints

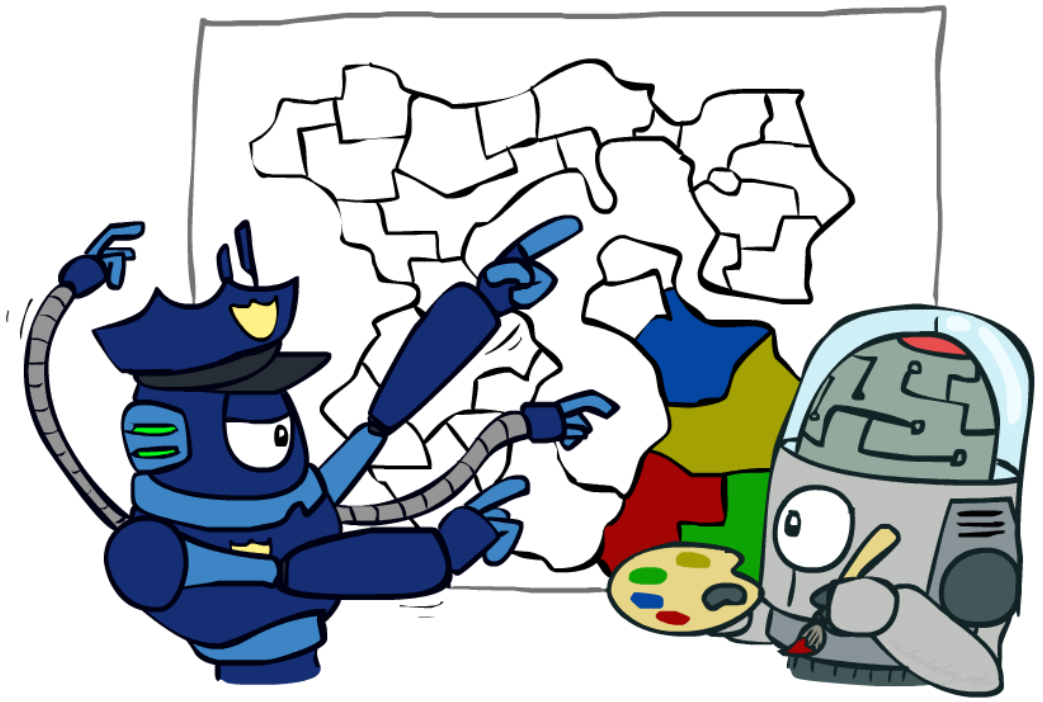
- Unary constraints involve a single variable (equivalent to reducing domains), e.g.:

$$SA \neq green$$

- Binary constraints involve pairs of variables, e.g.:

$$SA \neq WA$$

- *Higher-order* constraints involve 3 or more variables:
  e.g., cryptarithmetic column constraints

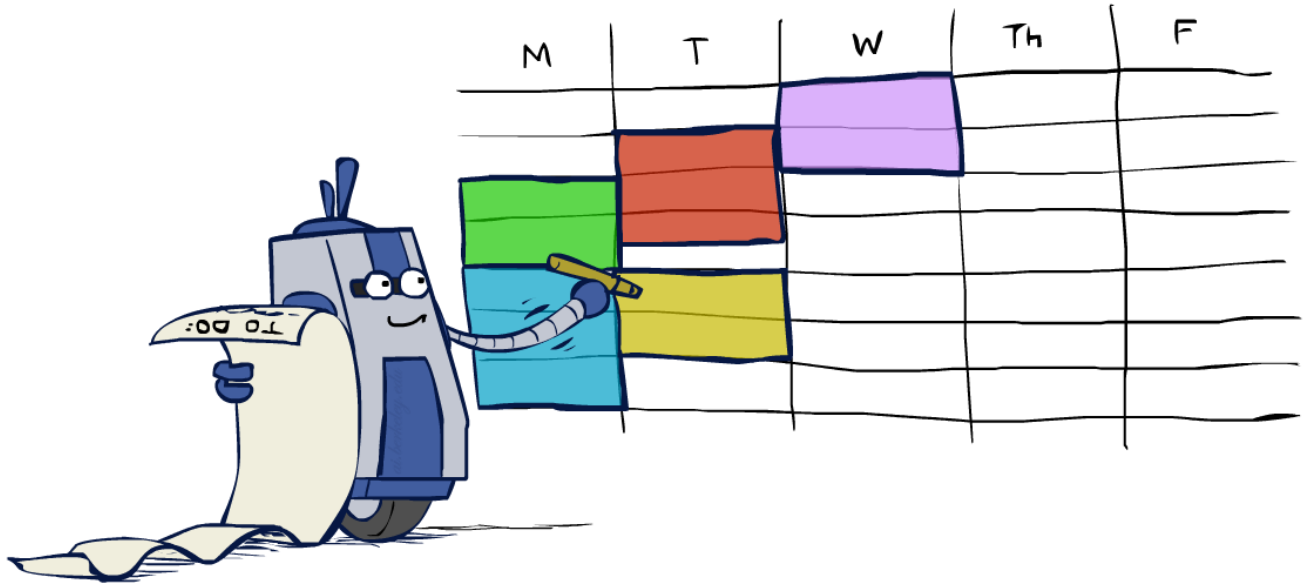


Preferences (soft constraints):

- E.g., red is better than green
- Often representable by a cost for each variable assignment
- Gives constrained optimization problems
- (We'll ignore these until we get to Bayes' nets)

Real-world CSPs

- Assignment problems: e.g., who teaches what class
- Timetabling problems: e.g., which class is offered when and where?
- Hardware configuration
- Transportation scheduling
- Factory scheduling
- Circuit layout
- Fault diagnosis
- … lots more!



- Many real-world problems involve real-valued variables…

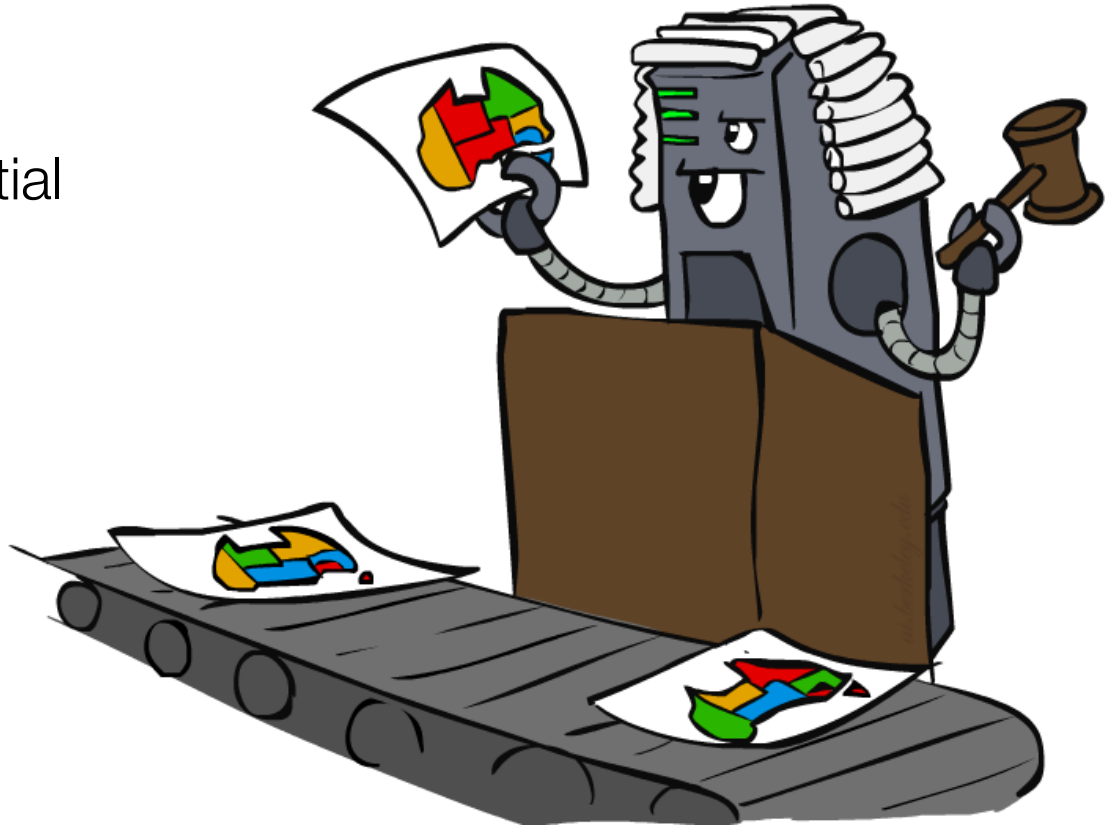
Great, but how do we solve these things?

Standard search formulation

Standard search formulation of CSPs

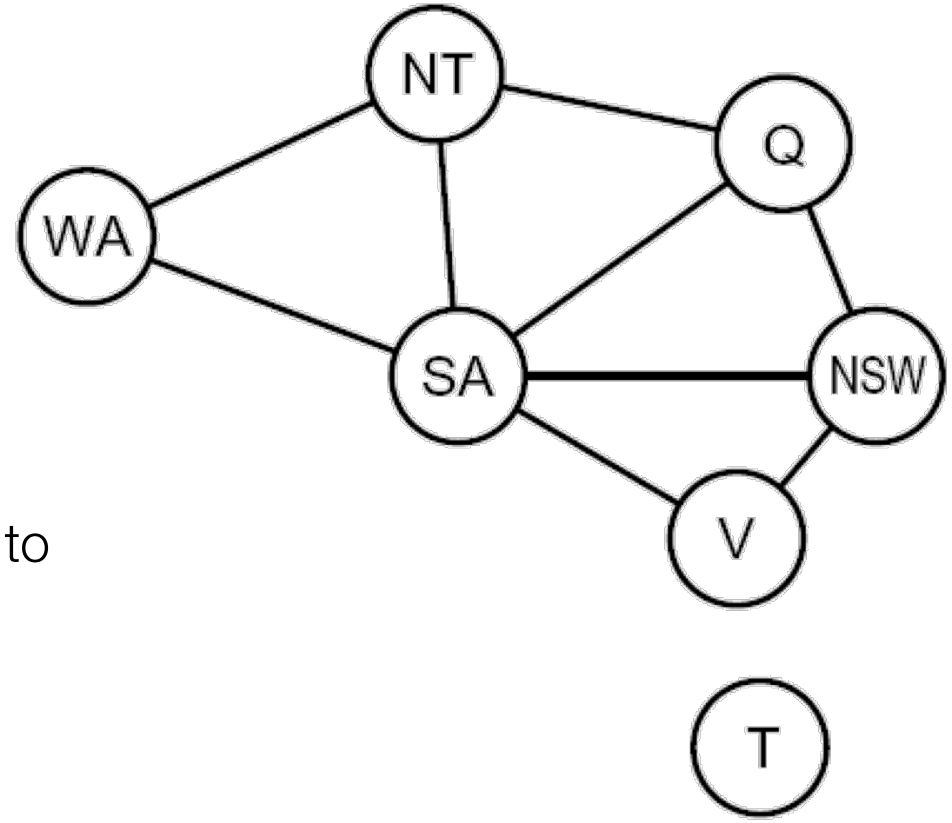
States defined by the values assigned so far (partial assignments)

- Initial state: the empty assignment, {}
- Successor function: assign a value to an unassigned variable
- Goal test: the current assignment is complete and satisfies all constraints



In-class exercise

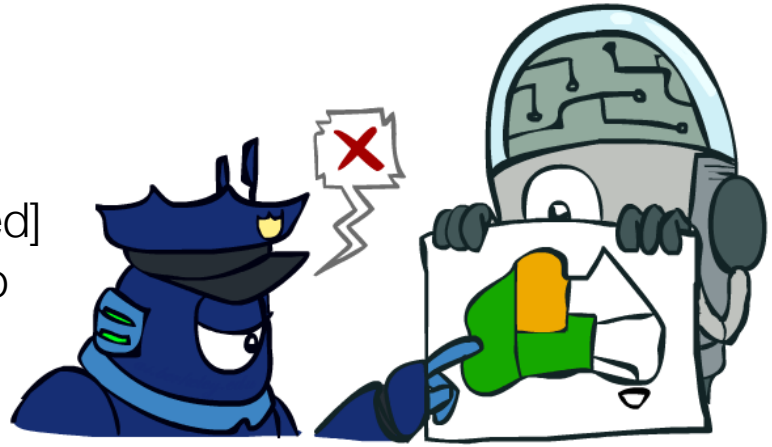
- Work on your own or in small groups
- Be sure to write everyone's name *legibly*!
- Oops! Hand-out should read "*successor function* to generate children is



How’d we do?

Backtracking search

Backtracking search is the basic uninformed algorithm for solving CSPs



Idea 1: *One variable at a time*

- Variable assignments are commutative, so fix ordering
- I.e., [WA = red then NT = green] same as [NT = green then WA = red]
- Only need to consider assignments to a single variable at each step

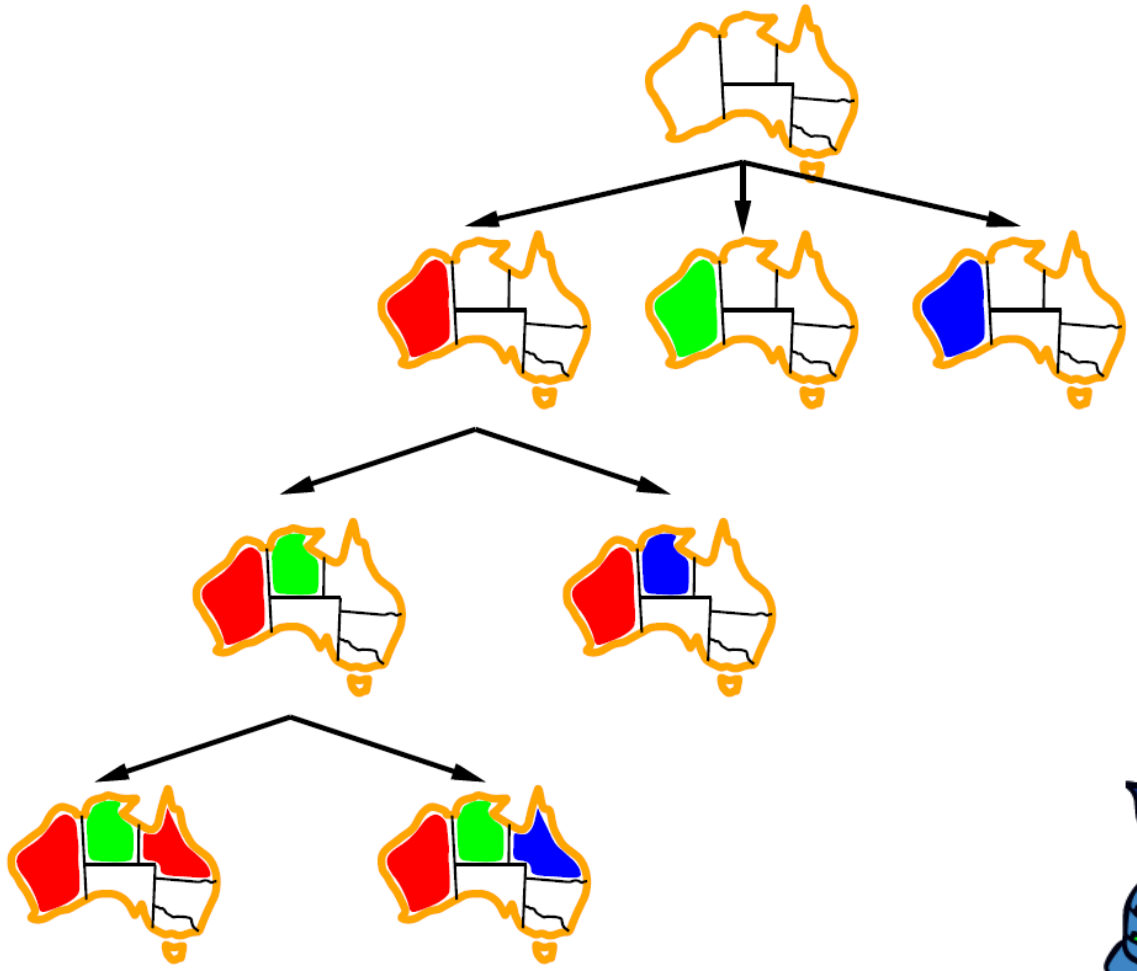
Idea 2: *Check constraints as you go*

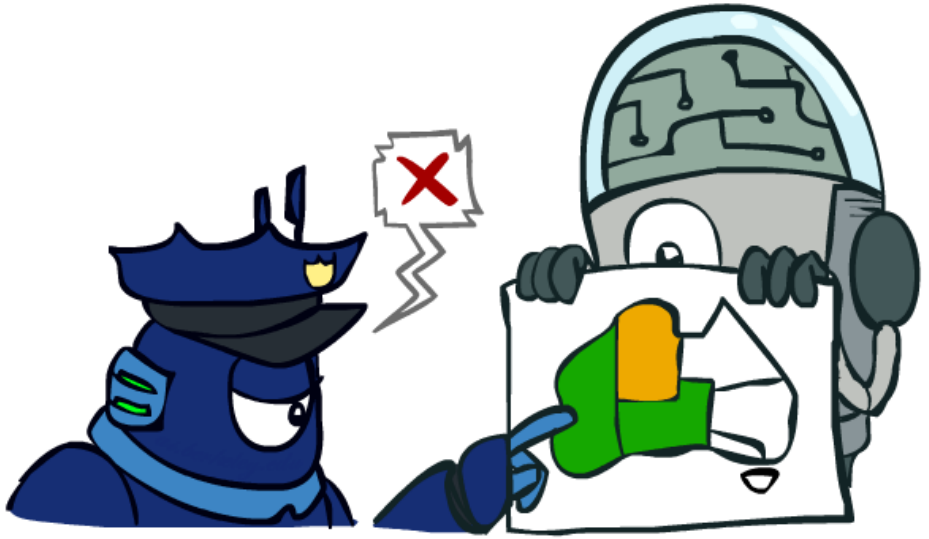
- I.e. consider only values which do not conflict previous assignments
- Might have to do some computation to check the constraints
- “Incremental goal test”

Depth-first search with these two improvements is called *backtracking search* (not the best name)

Can solve n-queens for n ≈ 25

Backtracking example





Backtracking search

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
    return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment given CONSTRAINTS[csp] then
            add {var = value} to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failure then return result
            remove {var = value} from assignment
    return failure
```

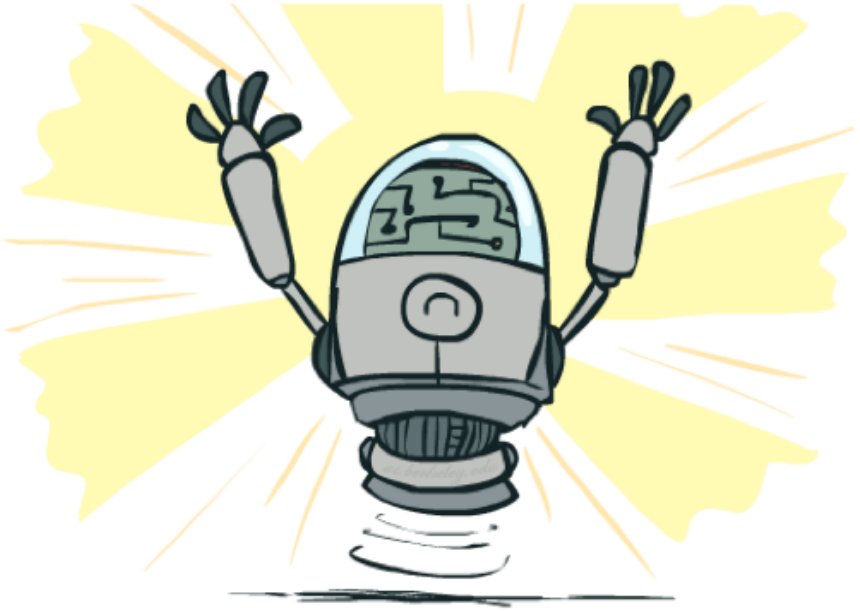
Backtracking = DFS + variable-ordering + fail-on-violation

Improving backtracking

Ordering:

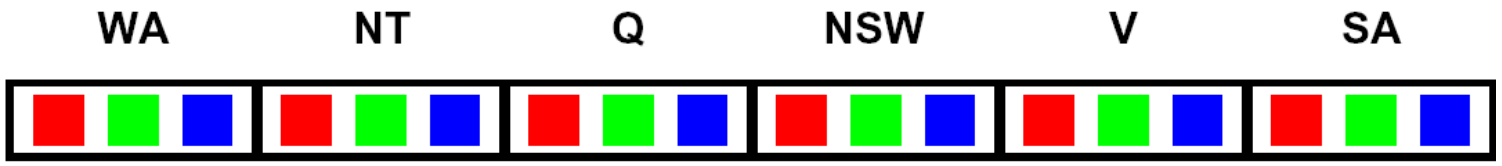
- Which variable should be assigned (expanded) next?
- In what order should its values be tried?

Filtering: Can we detect inevitable failure early?

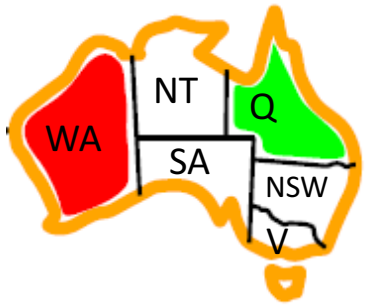


Filtering: forward checking

Filtering: Keep track of domains for unassigned variables and cross off bad option



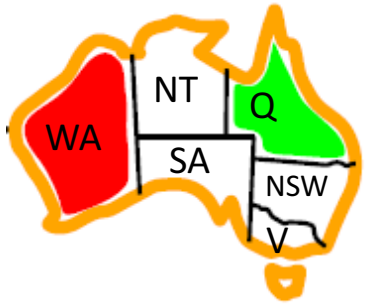
Forward checking

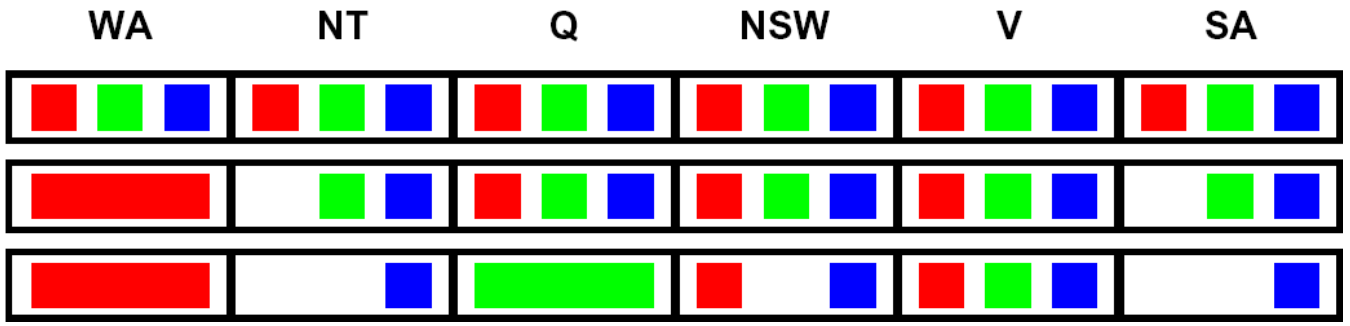


Assume we have assigned WA red. Let's think about the remaining vars.

Filtering: constraint propagation

Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:

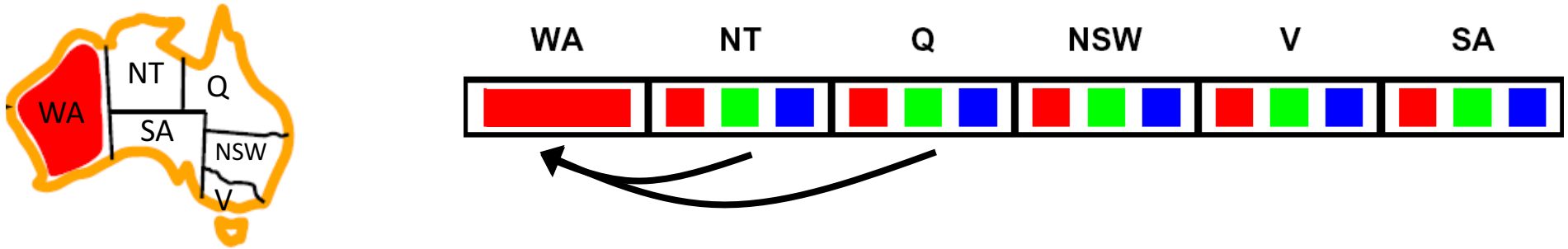




- NT and SA cannot both be blue!
- Why didn't we detect this yet?
- *Constraint propagation:* reason from constraint to constraint

Consistency of a single arc

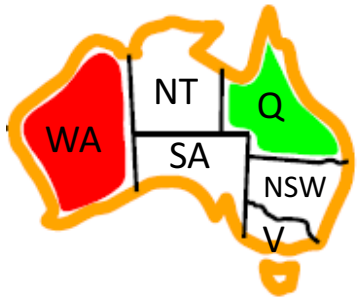
An arc X → Y is **consistent** iff for *every* x in the tail there is *some* y in the head which could be assigned without violating a constraint

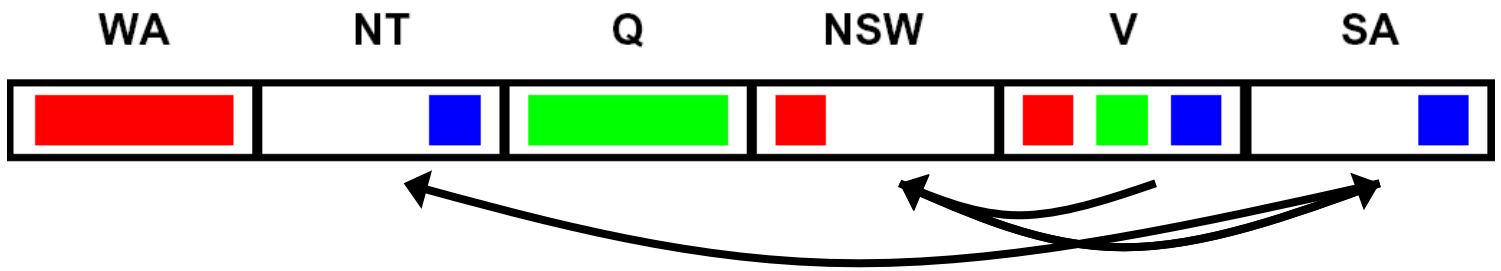


Forward checking: Enforcing consistency of arcs pointing to each new assignment

Arc consistency of an entire CSP

A simple form of propagation makes sure **all** arcs are consistent:





- **Important**: *If X loses a value, neighbors of X need to be rechecked*!
  - i.e., you need to reconsider all arcs going into!
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment
- What's the downside of enforcing arc consistency?

Remember:
Delete from the tail!

Enforcing arc consistency in a CSP

function AC-3(*csp*) **returns** the CSP, possibly with reduced domains
  inputs: *csp*, a binary CSP with variables $\{X_1, X_2, \ldots, X_n\}$
  local variables: *queue*, a queue of arcs, initially all the arcs in *csp*

  while *queue* is not empty **do**
    $(X_i, X_j) \leftarrow$ REMOVE-FIRST(*queue*)
    if REMOVE-INCONSISTENT-VALUES(X_i, X_j) **then**
      for each X_k **in** NEIGHBORS$[X_i]$ **do**
        add (X_k, X_i) to *queue*

---

function REMOVE-INCONSISTENT-VALUES(X_i, X_j) **returns** true iff succeeds
  removed $\leftarrow$ *false*
  for each x **in** DOMAIN$[X_i]$ **do**
    if no value y in DOMAIN$[X_j]$ allows (x,y) to satisfy the constraint $X_i \leftrightarrow X_j$
      then delete x from DOMAIN$[X_i]$; *removed* $\leftarrow$ *true*
  return *removed*

Enforcing arc consistency in a CSP

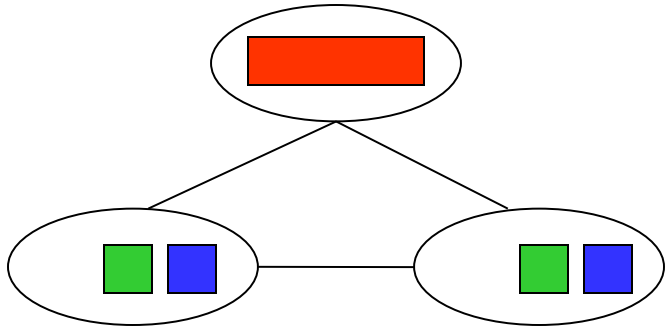
```
function AC-3( csp) returns the CSP, possibly with reduced domains
  inputs: csp, a binary CSP with variables {X_1, X_2, ..., X_n}
  local variables: queue, a queue of arcs, initially all the arcs in csp

  while queue is not empty do
    (X_i, X_j) ← REMOVE-FIRST(queue)
    if REMOVE-INCONSISTENT-VALUES(X_i, X_j) then
      for each X_k in NEIGHBORS[X_i] do
        add (X_k, X_i) to queue

---------------------------------------------------------------------

function REMOVE-INCONSISTENT-VALUES( X_i, X_j) returns true iff succeeds
  removed ← false
  for each x in DOMAIN[X_i] do
    if no value y in DOMAIN[X_j] allows (x,y) to satisfy the constraint X_i ↔ X_j
      then delete x from DOMAIN[X_i]; removed ← true
  return removed
```

- Naïve runtime: $O(n^2d^3)$ -- can be reduced to $O(n^2d^2)$
- … but detecting all possible future problems is NP-hard – why?

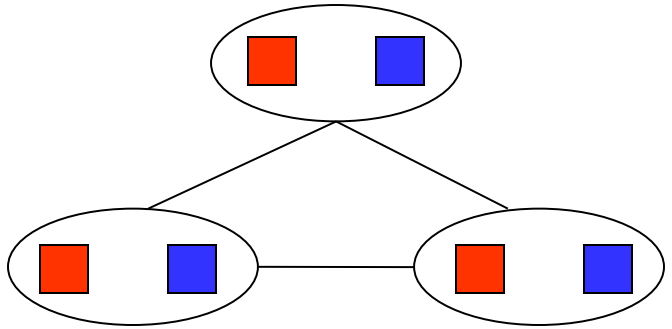
Limitations of arc consistency

After enforcing arc consistency:

- Can have one solution left
- Can have multiple solutions left
- Can have no solutions left (and not know it)



Arc consistency still runs inside a backtracking search!



What went
wrong here?

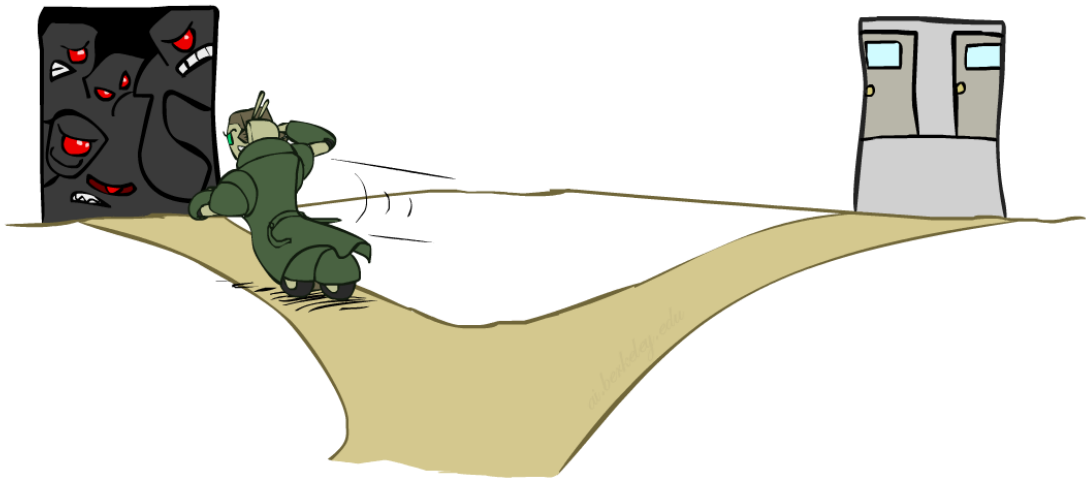
Ordering

Ordering: minimum remaining values

Variable Ordering: Minimum remaining values (MRV):

- Choose the variable with the fewest legal left values in its domain

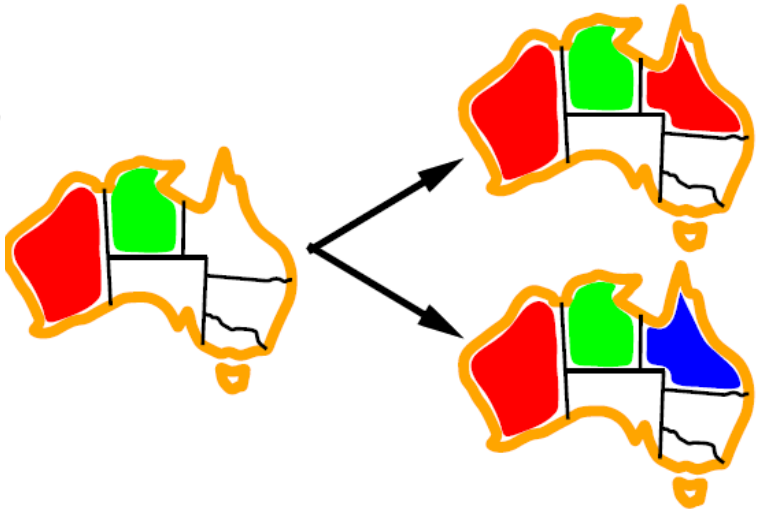
- Why min rather than max?
- Also called “most constrained variable”
- “Fail-fast” ordering



Ordering: least constraining value

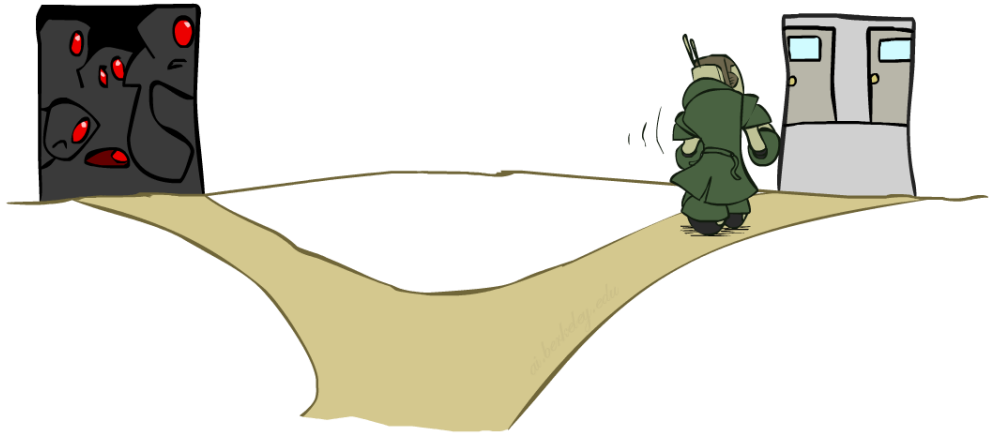
Value Ordering: *Least Constraining Value*

- Given a choice of variable, choose the *least constraining value*
- I.e., the one that rules out the fewest values in the remaining variables
- Note that it may take some computation to determine this! (E.g., rerunning filtering)



Why least rather than most?

Combining these ordering ideas makes 1000 queens feasible



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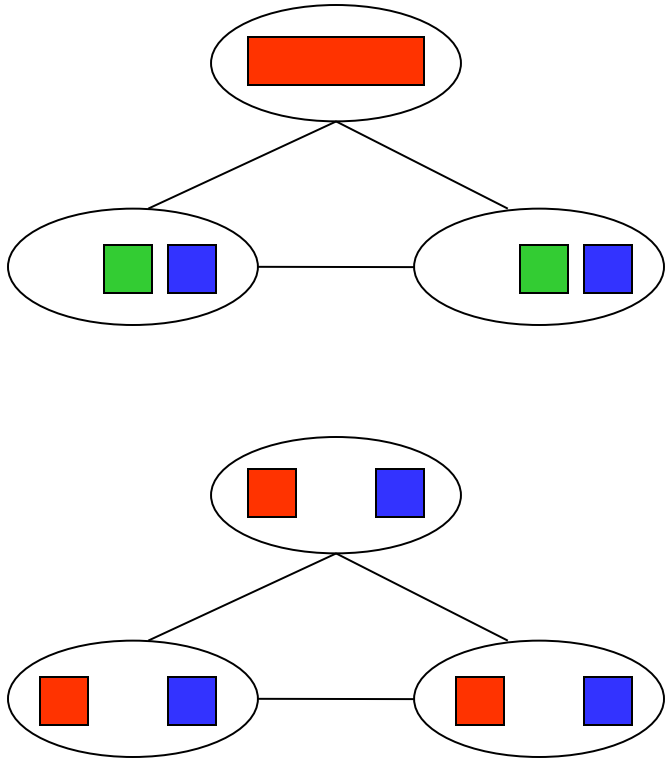
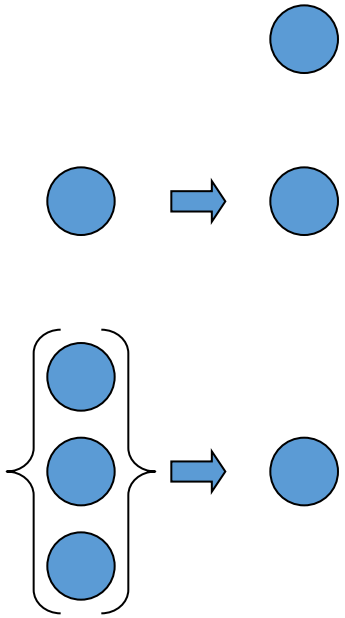


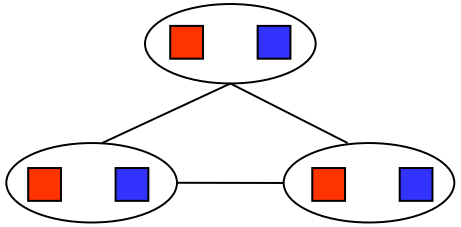
What went wrong here?

K-consistency: beyond pairs

Increasing degrees of consistency

- 1-Consistency (Node Consistency): Each single node's domain has a value which meets that node's unary constraints
- 2-Consistency (Arc Consistency): For each pair of nodes, any consistent assignment to one *can* be extended to the other
- *K*-Consistency: For each *k* nodes, any consistent assignment to k-1 *can* be extended to the k^{th} node.





Higher *k* more expensive to compute

(You need to know the *k*=2 case: arc consistency)

Q

Can a CSP be *k*-consistent but not *k*-1 consistent?

Strong *K*-consistency

- Strong *k*-consistency: also k-1, k-2, … 1 consistent
- Claim: strong n-consistency means we can solve without backtracking!
- Why?
  - Choose any assignment to any variable
  - Choose a new variable
  - By 2-consistency, there is a choice consistent with the first
  - Choose a new variable
  - By 3-consistency, there is a choice consistent with the first 2
  - …
- Lots of middle ground between arc consistency and n-consistency! (e.g. k=3, called path consistency)

Exploiting structure

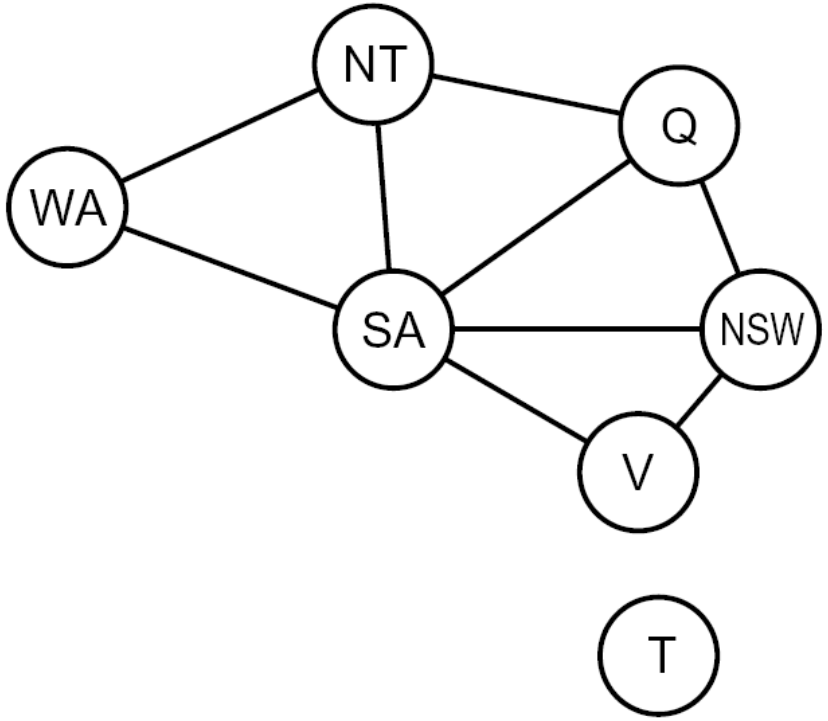
Extreme case: independent subproblems

- Example: Tasmania and mainland do not interact

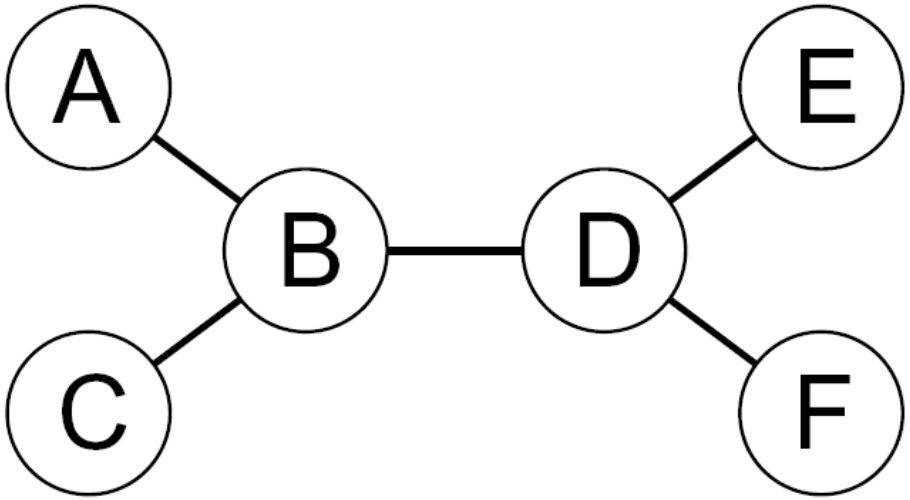
Independent subproblems are identifiable as connected components of constraint graph

Suppose a graph of n variables can be broken into subproblems of only c variables:

- Worst-case solution cost is $O((n/c)(d^c))$, linear in n
- E.g., n = 80, d = 2, c =20 (so 4 problems of size 20)
- 2^{80} = 4 billion years at 10 million nodes/sec
- $(4)(2^{20})$ = 0.4 seconds at 10 million nodes/sec



Tree-structured CSPs



Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n\ d^2)$ time

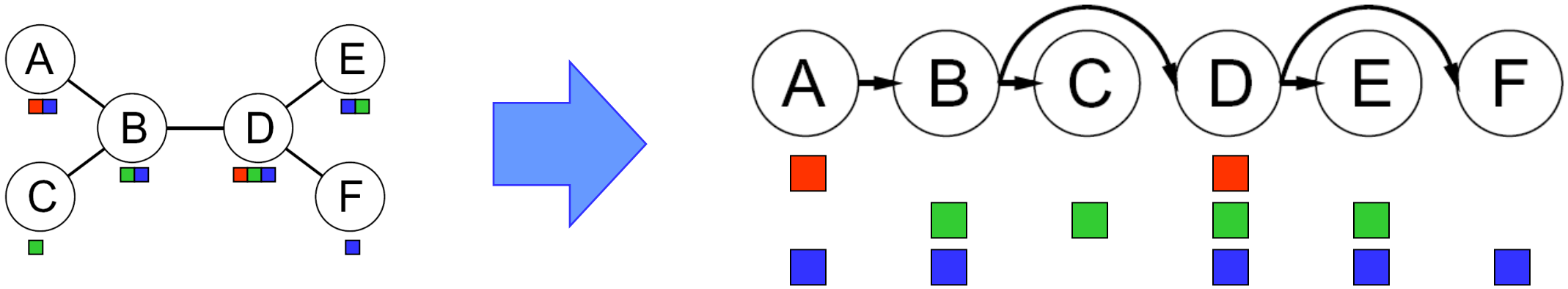
- Compare to general CSPs, where worst-case time is $O(d^n)$

This property also applies to probabilistic reasoning (later): an example of the relation between syntactic restrictions and the complexity of reasoning

Tree-structured CSPs

Algorithm for tree-structured CSPs:

Order: Choose a root variable, order variables so that parents precede children



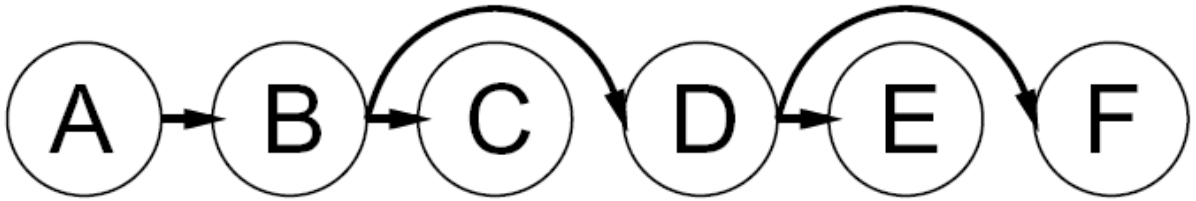
Remove backward: For i = n : 2, apply RemoveInconsistent(Parent(X_i),X_i)
Assign forward: For i = 1 : n, assign X_i consistently with Parent(X_i)

Runtime: O(n d^2) (why?)

Tree-structured CSPs

Claim 1: After backward pass, all root-to-leaf arcs are consistent

Proof: Each X→Y was made consistent at one point and Y's domain could not have been reduced thereafter (because Y's children were processed before Y)



Claim 2: If root-to-leaf arcs are consistent, forward assignment will not backtrack

Proof: Induction on position

How does this rely on the tree structure again?

Note: we'll see this basic idea again with Bayes' nets

Summary: CSPs

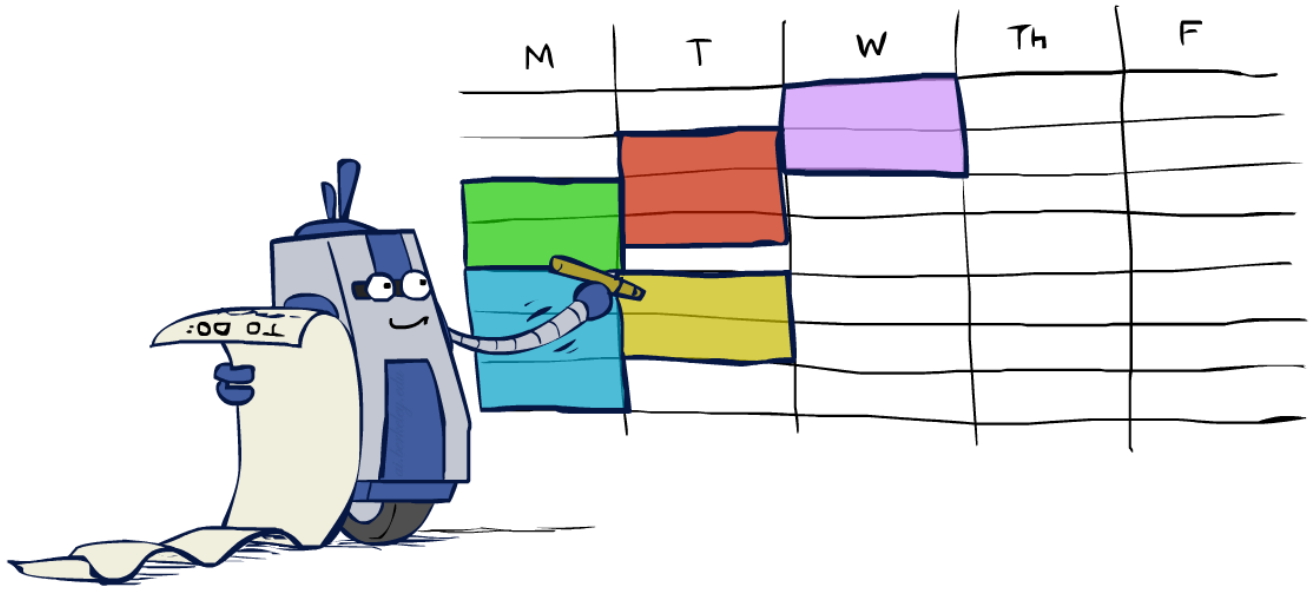
CSPs are a special kind of search problem:

- States are partial assignments
- Goal test defined by constraints

Basic solution: backtracking search

Speed-ups:

- Ordering
- Filtering
- Structure



Iterative min-conflicts is often effective in practice

That's it for today!

Next time: *adversarial search*